Title: Multiple Representations of Limits

Brief Overview:

Using graphical, numerical, and algebraic approaches, Pre-Calculus students develop an intuitive understanding of the concept of the limit in preparation for a more rigorous treatment in Calculus.

Links to Standards:

- **Mathematics as Problem Solving**
  Students use a variety of problem solving approaches in evaluating limits and select the most effective and appropriate method.

- **Mathematics as Communication**
  Students will develop and demonstrate their understanding of limits through writing on limits in a variety of contexts.

- **Mathematics as Reasoning**
  Students will be able to choose between different problem solving approaches. Students will check and test their conjectures through multiple representations of limits. Students also use inductive reasoning in evaluating limits numerically. Developing and articulating an understanding of the definition of a limit will require sophisticated reasoning.

- **Mathematical Connections**
  Students develop an understanding of the connections between functions, graphs, and sequences through their evaluation of limits through multiple representations.

- **Algebra**
  Students use algebraic techniques in evaluating limits and appreciate the efficiency of these techniques in certain contexts.

- **Functions**
  Students work with functions graphically, analytically, and numerically through tabular analysis.

- **Conceptual Underpinnings of Calculus**
  Students develop an intuitive understanding of the limit of a function, the basis of differential and integral calculus.

Links to Maryland High School Mathematics Core Learning Goals:

This learning unit encompasses the first half of Goal 1: Functions and Algebra, involving the analysis of patterns and functions using the language of mathematics and appropriate technology. The following Indicators are specifically addressed:

- **1.1.1:** Students recognize patterns, describe, and extend patterns and functional relationships that are expressed numerically, algebraically, and graphically.

- **1.1.2:** Students represent patterns and functional relationships in a table, as a graph, and by mathematical expressions.

- **1.1.3:** Students perform operations with algebraic functions.

- **1.1.4:** Students describe the graph of non-linear functions.

Grade/Level:

Pre-Calculus (Grades 11/12).

Prerequisite Knowledge:

Students should have working knowledge of the following skills:

- Evaluation and graphing of Pre-Calculus level functions, including polynomials, trigonometric functions, rational functions, greatest integer function, and piecewise functions
- Second year algebra skills such as factoring, adding rational expressions, and conjugate multiplication
- On the TI-83, construct a graph in an arbitrary viewing window; construct a table; piecewise graphing.

Objectives:

Students will:

- develop an intuitive understanding of what a limit is.
- be able to evaluate limits graphically, numerically, and algebraically.
- demonstrate their understanding through writing on limits.

Materials/Resources/Printed Materials:

• TI-83 Graphics Calculator for each student or student group

Development/Procedures:

In an approximately week-long project (five traditional length periods) students will explore limits, learn how to evaluate them, and develop a strong intuitive understanding of them.

A) Students encounter limits initially through "The Bad Luck Ladder" application problem.

B) Students evaluate a set of limits graphically.

C) Students evaluate a set of limits numerically with tables.

D) Students use three different algebraic techniques (factoring, conjugate multiplication, and addition of rational expressions) to evaluate limits. Examples of each technique should be given by the instructor beforehand.

E) Students demonstrate their understanding through a writing exercise on limits.

Performance Assessment:

Students will be assessed on the basis of their completion of the project, including evaluation of limits by all techniques, comprehension questions, and writing assessments. In addition, the instructor may wish to add a traditional quiz in which the students evaluate limits of the types encountered in these Activities with all techniques studied.

Extension/Follow Up:

A number of follow-ups are appropriate for this lesson.

1. Rigorous: Connect results of this lesson to the formal epsilon-delta definition of limit.

2. Intuitive: Pose the problem

$$\lim_{x \to +\infty} \frac{x+1}{x-3}$$

Ask students to determine the answer to this question using the numerical and graphical methods of this lesson.

3. In-Class Project: On graph paper students graph $y = x^2$ from $x = 0$ to $x = 5$. By using rectangles, approximate the area under the curve. Experiment with increasing the number of rectangles. Relate this activity to limits.

4. Graphing Calculator: Explore local linearity.

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Activity A: The Bad Luck Ladder

One day a friendly math teacher was busily painting the shutters on the second floor of his home, carefully perched on the top of a 13' ladder. Hard to believe, but as he was finishing the last shutter, a mischievous student passed his house and, grabbing the base of the ladder, began to pull the base away from the house at a constant rate of 1 foot/second. The top of the ladder then began to head to the ground (the teacher immediately realized!) at the rate of $\dfrac{-x}{\sqrt{169 - x^2}}$ feet per second, where x is the distance from the base of the house to the base of the ladder.

1. Draw a picture of this situation.

2. Evaluate the rates at x =1, x=5, x=8.

3. What is the significance of the sign of the answers?

4. What is the domain for this problem? Explain.

5. Graph the rate function on your calculator. Use the ZOOM Box function to investigate the behavior of the graph as x approaches 13.

6. Is it possible to answer the question: With what rate does the poor math teacher crash to the ground?

The above exercise serves as an introduction to limit theory. Limit theory, a critical element in the development of calculus, revolves around the question: As x approaches a certain number (or possibly +/- infinity) what behavior does f(x) exhibit? The symbolism for limit is

$$\lim_{x \to a} f(x) = L$$

which is read "the limit as x approaches a of f(x) is L."

In the following Activities you will learn techniques for evaluating limits and develop a strong intuitive understanding of just what a limit is. Good luck!

Activity B: Approaching Limits Graphically

In this activity you will evaluate limits like the one in Activity A using graphs produced on your graphing calculator.

1. Using your graphing calculator graph $f(x) = x^3 - 3x^2 - 1$. Use a Standard window.
   TRACE to find the value of f(x) as x approaches 2.

   ZOOM BOX several times to improve your evaluation.

2. Now using this graphing approach evaluate

   $$\lim_{x \to 2} \sin x$$

   Compare this limit to $\sin \dfrac{}{2}$. Compare the limit in Question 1 to f(2). Do you think that you can always evaluate limits by substituting the function value? Explain why or give a counterexample.

3. Evaluate the following limits using a graphing approach:

   $$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

   $$\lim_{x \to 0} \frac{\sin x}{x}$$

   $$\lim_{x \to 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x}$$

   $$\lim_{x \to 1} \frac{|x - 1|}{x - 1}$$

4. How might you evaluate limits without graphing?

Activity C : Limits through Numerical Analysis with Tables

This activity will make use of the TABLE function on your calculator to evaluate limits.

1. Evaluate the limit

$$\lim_{x \to 2} x^4 - 3x^2 + 1$$

by using the TABLE function on your calculator.

Set Tbl Start at 1 and delta table at .1.

```
TABLE SETUP
 TblStart=1
 △Tbl=.1
Indpnt: AUTO Ask
Depend: AUTO Ask
```

```
X        Y1
1.300   ⁻1.214
1.400   ⁻1.038
1.500   ⁻.688
1.600   ⁻.126
1.700    .682
1.800    1.778
1.900    3.202
X=1.9
```

a) **In chart form list the three x values closest to x = 2 with their corresponding f(x) [Y1] value.**

      The chart for the table listed above would look like this.
      (one blank chart is included as a guide for further exercises)

          DELTA .1
        x   **f(x)**

x	f(x)
1.7	0.682
1.8	1.778
1.9	3.202

DELTA _____

x	f(x)

b) **Construct additional charts for delta equal to .05, .001, .005.**
From the charts identify the limit as x approaches 2 of f(x).

c) **Repeat the charting exercise approaching x = 2 from x = 3 (that is, the right**
side) using deltas -0.1,-0.05, -0.001,-0.005.

d) Is it important to construct a table approaching 2 from both 1 and 3?

2. Construct Tables as in the above exercise to evaluate the following limit -
   (use the same delta values)

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

a) Does the limit you identified equal f(3)?

b) Must f(3) exist in order for the limit to exist ?

3. Construct Tables to evaluate the following limits.

$$\lim_{x \to 4} \frac{5}{x-4}$$

$$\lim_{x \to 3} \frac{-5}{(x-4)^2}$$

a) Discuss your results. How are they the same or different ?

For a function to have a limit is to imply that the limit is a real number. Infinity, positive infinity, and negative infinity are not real numbers and are not considered limits. Sometimes the symbols

$+\infty$ (positive infinity)   and $-\infty$ (negative infinity)

are used in limit statements since they describe the behavior of f(x).

$$f(x) = \begin{cases} x^2 - 6, x \le -1 \\ -x + 5, x > -1 \end{cases}$$

4. Consider:

a)   Using tables evaluate the limit of f(x) as x approaches -1. Be sure to approach from the
      -2 side of -1 and from the 0 side of -1.

b)   Does the limit at -1 exist? Explain.

c)   Prepare a graph of the function using the piecewise graphing capabilities of your
      calculator.

5. Graph the greatest integer function, $f(x) = [x]$.

     NOTES :

     i) The greatest integer function of x is the largest integer less than or equal to x.
     ii) Use int(x) on your calculator to graph this function (located in MATH under NUM).
     iii) Use DOT graphing when graphing this function.

a)   What is interesting about this graph?

b)   What values of x will have limits and what values will not have limits?

c)   Create a table for this function and consider approaching x=2 from 1 (the left of 2)

and then from 3 (the right of 2).

Since limits describe the behavior of a function, it is useful to talk about the result of an approach only from the left of x = a or an approach only from the right of x = a. An approach from the left is called a left hand limit and one from the right is called a right hand limit.

The symbolism is

$$\lim_{x \to a^-} f(x)$$

$$\lim_{x \to a^+} f(x)$$

for a left hand limit and for a right hand limit respectively.

The +/- symbols used in these approaches indicate the direction in which the x's approach a and do not imply the sign of the numbers used in the approach.

6. Identify a function where the limit as x approaches 3 from the left does not exist but can be

described with the indicator +   .

Activity D: Evaluating Limits Algebraically

Many limits can be evaluated algebraically without tables or graphs.

1. Find these limits using only algebraic techniques.

$$\lim_{x \to 4} \quad 3x + 1$$

$$\lim_{x \to 2} \quad \frac{x - 2}{x^2 - 4}$$

$$\lim_{x \to 7} \quad \frac{2x^2 - 11x - 21}{x - 7}$$

$$\lim_{x \to -3} \quad \frac{x^3 + 27}{x + 3}$$

$$\lim_{x \to 0} \quad \frac{\sqrt{x + 3} - \sqrt{3}}{x}$$

2. Note that whereas direct substitution into the function yields the limit in the first problem above, it results in 0/0 in all of the others. That is why algebraic steps such as factoring are necessary before substituting. The form 0/0 is called an "indeterminate form."

a. Why do you think 0/0 is called an indeterminate form?

b. Can you find another indeterminate form?

c. Is 5/0 an indeterminate form?

d. Evaluate

$$\lim_{x \to 7} \quad \frac{1}{x - 7}$$

3. Evaluate the following limits algebraically:

$$\lim_{h \to 0} \frac{3(x+h) + 1 - (3x+1)}{h}$$

$$\lim_{h \to 0} \frac{2(x+h)^2 + 5 - (2x^2 + 5)}{h}$$

$$\lim_{h \to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h}$$

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

4. For what types of limit problems is the algebraic technique more efficient?

5. Give examples where graphs or tables are needed to clarify the answer to a limit problem.

Activity E: Writing about Limits

In this activity you will think about limits, discuss them with your classmates, and write at least one good paragraph relating what you now know about limits to each of the following situations.

1. You are driving with you grandmother, telling her about what you are learning at school. She tartly remarks that you don't seem to know much about *speed* limits!

2. As you leave the house, your father asks, "Where are you going?" "To the library, sir!" you answer." You do pass the library on the way to the arcade.

3. When baby-sitting for a pair of rambunctious four year old twins, you firmly instruct them to play CLOSE TO the house. There is repeated discussion about what this means.

4. Your mother asks you to pick up your little sister from practice at 5:00. You pick her up at 6:00.

5. You just LOVE gravy with your turkey at Thanksgiving. But after proceeding around the crowded holiday table, the gravy boat is empty when it gets to you! Mom refills it in the kitchen and sends it around again, but by the time it reaches you it is empty once more!

6. Susan, Pablo, and Tony each drop a ball at the same time off of a second-story balcony as part of an informal experiment. Tony drops an old tennis ball, Pablo a ball made of putty, and Susan a "super"-ball that is perfectly elastic (imagine that!).

Notes and Suggestions for the Instructor

1. The expectation is that students work through these Activities at the end of a Honors level Pre-Calculus class or, perhaps, at the start of a Calculus class. The Activities are intended to be done almost entirely by the students, but more guidance from the instructor may be appropriate in certain sections.

2. In Activity A, the rate of fall of the top of the ladder is derived as follows: Let x be the distance between the base of the ladder and the base of the house and let y be the distance between the top of the ladder and the base of the house. Then x and y are related by the Pythagorean Theorem: $x^2 + y^2 = 13^2$. Differentiating with respect to t (time) yields $2x(dx/dt) + 2y (dy/dt) = 0$. Now solving for the rate of fall of the top of the ladder gives $dy/dt = (-x/y)(dx/dt)$. Solving for y in the Pythagorean equation above and substituting that expression in terms of x into this equation, and also substituting the rate $dx/dt = 1$ which is given in the problem, yields the equation for dy/dt given in the Activity.

3. In Activities B, C, and D the instructor may well wish to assign more limit problems to give the students more opportunities for practice and learning; these can be found in Chapter 1 of almost any Calculus textbook.

4. In Activity D, the algebraic approach, students who do not have strong algebra backgrounds may need to have an example of the different techniques worked for them first. This could be done at the board by the instructor before the students begin work on that activity.

5. In Activity E, the writing section, the connections with limits have deliberately been left implicit. The students should understand that thoughtful answers are expected! And we strongly suggest that the instructors provide NO HINTS at this point!